MATHEMATICS

# ON BISHOP'S GENERALIZATION OF THE WEIERSTRASS-STONE THEOREM

BY

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## § 1. INTRODUCTION

We here present a proof of the main result of Bishop [1] restated, for convenience, in § 4 below, in which:

i) no measure theory is used;
ii) Zorn's Lemma is directly used one time as the only transfinite tool coming into play.

We assume given the partition into maximal antisymmetric sets directly built as in Glicksberg [4].

Actually, we consider continuous normed space valued functions on a compact Hausdorff space and prove a module version of strong type of Bishop's theorem for such functions. This "strong" terminology is due to Buck [3]. It refers to the fact that one computes the distance of an arbitrary continuous function from a given module or algebra of analogous functions. This is done in terms of the respective distances, from the restrictions of the given function to the maximal antisymmetric sets, to the sets of the corresponding restrictions of the elements in the module – a mini-max formula.

In the context of Bishop's theorem, and for scalar valued functions, the types of result we are referring to are due to Glicksberg [4]: a strong form for algebras in page 419 and a module result in a remark in page 434. It seems that Prolla [10] is the first to establish a vector-valued extension of Bishop's theorem. There, in fact, Prolla is concerned with the more difficult problem of weighted approximation in the sense of Nachbin [8], [9] in the non self-adjoint complex case; his methods are inspired by the ones of Glicksberg [4].

The present methods expose Bishop's theorem as an equivalent formulation of the classical Weierstrass theorem on polynomial approximation – modulo Zorn's Lemma and elementary topological arguments. In spirit we remain very close to Bishop's reasoning in [1]. Theorem 1

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and Corollary 1 below, standard Weierstrass-Stone theorems in strong form, are very precise versions of Bishop's technical lemma in [1]. The unavoidable "hard analysis" detail occurs in the proof of Lemma 1 below. It is the cleverly proved Lemma 2 in Jewett [5]. Lemma 1 is a variant of the lemma in § 5 of Nachbin [8]. The proof of Theorem 2 in Buck [3] was also instrumental.

## § 2. BASIC DATA AND BASIC LEMMA

We suppose given a non-empty compact Hausdorff space $X$ and a normed space $E$ with norm $\|\cdot\|$; $E$ is a vector space over $K$ where $K$ is either the real field $R$ or the complex field $C$. The vector space $C(X;E)$ of all continuous $E$-valued functions on $X$ is provided with the uniform norm and we may have $E=K$.

$A$ denotes a subalgebra of $C(X;K)$ containing the constants and $A(X)$ is the collection of the maximal sets of anti-symmetry for $A$ (Glicksberg [4]).

REMARK 1: In case $K=R$, or if $K=C$ and $A$ is self-adjoint, $A(X)$ is, equivalently, the collection of the equivalence classes modulo the $A$-equivalence relation on $X$: $x, y \in X$ are $A$-equivalent if $a(x)=a(y)$ for all $a \in A$.

REMARK 2: Suppose, for a moment, that $A$ is contained in $C(X;R)$. Then it is easy to see that each $Y \in A(X)$ is an intersection of $A$-peak sets (Browder [2], page 140). Therefore, if $D$ is a compact subset of $X$ disjoint from $Y$, there is $a \in A$ such that $0<a \leqslant 1$ with $a(x)=1$ for $x \in Y$ and $a(x)<1$ for $x \in D$ (Leibowitz [6], page 160). This remark is used in the proof of the basic Lemma 1 below.

REMARK 3: We shall work with vector subspaces $G$ of $C(X;E)$ which are modules over the algebra $A$: $AG \subset G$. Then $G$ is also a module over the set $B$ of the real elements in $A$ which is a subalgebra of $C(X;R)$ containing the constants. Fundamental use is made of the corresponding partition $B(X)$ of $X$ given by the $B$-equivalence relation on $X$.

LEMMA 1: For each $Y \in B(X)$ let there be given a compact subset $D_Y$ of $X$ disjoint from $Y$. Then, there are $Y_1, \ldots, Y_n$ in $B(X)$ such that: to each given $\delta>0$ there correspond $b_1, \ldots, b_n$ in $B$ such that

$$1-\delta<b_1+b_2+\ldots+b_n \leqslant 1, \; 0 \leqslant b_j \leqslant 1$$

and $b_j(x)<\delta$ for $x \in D_Y$ with $Y=Y_j$ $(j=1, \ldots, n)$.

PROOF: For each $Y \in B(X)$ choose $C_Y \in B$ such that $0 \leqslant C_Y \leqslant 1$ with $C_Y(x)=1$ for $x \in Y$ and $C_Y(x)<1$ for $x \in D_Y$. Then, $\sup\{C_Y(x);\, x \in D_Y\}<u_Y<1$ for some $u_Y$ and we may also pick $v_Y$ such that $0<u_Y<v_Y<1$. The set $Y$ is contained in the open set $U_Y=\{x \in X;\, C_Y(x)>v_Y\}$. By compactness, there are $Y_1, \ldots, Y_n$ in $B(X)$ such that $X=U_1 \cup \ldots \cup U_n$

where $U_j$ is $U_Y$ with $Y=Y_j$ $(j=1, \ldots, n)$; $D_j$ and $C_j$ are now, respectively, $D_Y$ and $C_Y$ with $Y=Y_j$ $(j=1, \ldots, n)$; analogously for $u_j$ and $v_j$.

Let $\delta>0$ be given. For each $j=1, \ldots, n$, apply Jewett [5], Lemma 2 to get a real polynomial $q_j$ on $[0,1]$ such that: $0 \leqslant q_j \leqslant 1$, $0 \leqslant q_j(t)<\delta$ for $0 \leqslant t \leqslant u_j$ and $1 \geqslant q_j(t)>1-\delta$ for $v_j \leqslant t \leqslant 1$. Put $g_j=q_j(C_j)$, $j=1, \ldots, n$.

The functions $g_1, \ldots, g_n$ are in $B$ and so are $b_1=g_1$, $b_2=(1-g_1)g_2, \ldots, b_n=$ $=(1-g_1) \ldots (1-g_{n-1})g_n$: this technique is from Rudin [11], item 2.13. It is easy to check by induction that $b_1+\ldots+b_n=1-(1-g_1) \ldots (1-g_n)$. Since $0 \leqslant g_j \leqslant 1$ and $0 \leqslant 1-g_j \leqslant 1$, there follows that $0 \leqslant b_j \leqslant 1$ and $0 \leqslant b_j(x) \leqslant g_j(x)<\delta$ for $x \in D_j$; $j=1, \ldots, n$. Now, for each $x \in X$, there exists $k(x)$ in $\{1, \ldots, n\}$ such that $x \in U_{k(x)}$. Then, $1 \geqslant g_{k(x)}(x)>1-\delta$ and

$$1 \geqslant (b_1+\ldots+b_n)(x)=1-[(1-g_{k(x)}) \textstyle\prod_{j \neq k(x)}(1-g_j)](x)>1-\delta.$$

The proof is complete.

## § 3. A STANDARD WEIERSTRASS-STONE THEOREM IN STRONG FORM

THEOREM 1: Let $G$ be a vector subspace of $C(X; E)$ which is a module over the unitary subalgebra $B$ of $C(X; R)$. Fix $f$ in $C(X; E)$. Then, $\inf\{\|f-g\|;\ g \in G\}=\sup\{\inf\{\|(f-g)|Y\|;\ g \in G\};\ Y \in B(X)\}$.

REMARK 4: Here $(f-g)|Y$ is the restriction of $f-g$ to $Y$ and $\|(f-g)|Y\|$ is the corresponding uniform norm. Notice the obvious formula:

$$\inf\{\|f-g\|; g \in G\}=\inf\{\sup\{\|(f-g)|Y\|;\ Y \in B(X)\}; g \in G\}.$$

PROOF: For each $Y \in B(X)$. Let $d(Y)=\inf\{\|(f-g)|Y\|; g \in G\}$ and put $d=\inf\{\|f-g\|; g \in G\}$ and $c=\sup\{d(Y);\ Y \in B(X)\}$. We must prove the inequality $d \leqslant c$ since it is clear that $d \geqslant c$. Let $\varepsilon>0$ be given. For each $Y \in B(X)$ choose $g_Y$ such that

$$d(Y)+\varepsilon>\|(f-g_Y)|Y\|=\sup\{\|f(x)-g_Y(x)\|; x \in Y\}.$$

The set $D_Y=\{x \in X;\ \|f(x)-g_Y(x)\| \geqslant d(Y)+\varepsilon\}$ is compact and disjoint from $Y$. With this data apply Lemma 1 with $\delta$ soon to be specified; change notation and call $g_j=g_Y$ and $D_j=D_Y$ with $Y=Y_j$, $j=1, \ldots, n$.

Choose $K>\|f\|+\|g_1\|+\ldots+\|g_n\|$ and let $\delta<(nK)^{-1}\varepsilon$.

Consider $g=b_1 g_1+\ldots+b_n g_n$ which belongs to $G$ because $G$ is a $B$-module. For each $x \in X$ we evaluate $\|f(x)-g(x)\| \leqslant e_1(x)+e_2(x)$ where

$$e_1(x)=\|f(x)-\sum_{j=1}^{n} b_j(x) f(x)\|$$

and

$$e_2(x)=\|\sum_{j=1}^{n} b_j(x)(f(x)-g_j(x))\|.$$

One gets $e_1(x) \leqslant\|f\| \cdot|1-(b_1(x)+\ldots+b_n(x))|<\delta\|f\|<\varepsilon$. Let

$$N(x)=\{j \in\{1, \ldots, n\};\ x \notin D_j\}$$

and $N'(x)=\{j \in\{1, \ldots, n\};\ x \in D_j\}$.

For $j \in N(x)$ it follows that $\|b_j(x)(f(x)-g_j(x))\| \leqslant (d(Y_j)+\varepsilon)\, b_j(x)$ while, for $j \in N'(x)$: $\|b_j(x)(f(x)-g_j(x))\| \leqslant \delta(\|f\|+\|g_j\|) < \delta K$. Therefore

$$e_2(x) \leqslant nK\delta + \sum_{j=1}^{n} (d(Y_j)+\varepsilon)\, b_j(x) \leqslant c+2\varepsilon.$$

We proved that $\|f-g\| \leqslant 3\varepsilon + c$, that is, $d \leqslant c+3\varepsilon$ which is all that is needed.

COROLLARY 1: Under the hypothesis of Theorem 1 there is $Y \in B(X)$ such that $\inf\{\|f-g\|;\, g \in G\} = \inf\{\|(f-g)\,|\,Y\|;\, g \in G\}$.

PROOF: Provide $B(X)$ with the quotient topology defined by the $B$-equivalence relation: one gets $B(X)$ as a compact Hausdorff space.
From [7], Lemma 1 (page 126), it follows that for each $g \in G$ the function $Y \mapsto \|(f-g)\,|\,Y\|$ is upper semicontinuous on $B(X)$. Therefore, the function $h$ on $B(X)$ defined by $h(Y) = \inf\{\|(f-g)\,|\,Y\|;\, g \in G\}$ is also upper-semicontinuous. There follows the existence of $Y(f) \in B(X)$ such that $h(Y(f)) = \sup\{h(Y);\, Y \in B(X)\}$ because $B(X)$ is compact, as we wanted.

REMARK 5: Under the hypothesis of Theorem 1, the function $f$ is in the closure of $G$ in $C(X; E)$ if, and only if, for any given $\varepsilon > 0$ and $Y \in B(X)$ there is $g \in G$ such that $\|f(x)-g(x)\| < \varepsilon$ for all $x \in Y$.

This is a usual form of the Weierstrass-Stone Theorem (Nachbin [9], § 19, Theorem 1). Having $G$ as a real or complex vector space, but keeping the elements of the algebra $B$ real, is useful in applications. When $G \subset C(X; K)$ is an algebra, $B$ is usually the set of the real valued elements of $G$ (see Remarks 1 and 3 in § 2); $B(X) = G(X)$ if $G$ is self-adjoint.
For instance, consider Theorem 3 in Rudin [12]. Functions $f, g_1, \ldots, g_n$ in $C(I; K)$, where I is the unit interval, are given, and $g_1, \ldots, g_n$ are real. $R(f, g_1, \ldots, g_n)$, the smallest closed subalgebra of $C(I, K)$ containing $f, g_1, \ldots, g_n$ and the constants, is assumed separating. The conclusion is that $R(f, g_1, \ldots, g_n) = C(I; K)$.
To obtain the conclusion as an application of Theorem 1 it is enough to prove that $R(f, g_1, \ldots, g_n)\,|\,Y$ is dense in $C(Y; K)$ for each $Y \in B(X)$, where $B$ is the unitary subalgebra of $C(I; R)$ generated by $g_1, \ldots, g_n$. Now, since $R(f, g_1, \ldots, g_n)$ is separating and the elements of $B$ are constant on $Y$, it follows that for distinct $x$ and $y$ in $Y$, $f(x) \neq f(y)$; that is, $Y$ and $f(Y)$ are homeomorphic. Any element $h$ in $C(Y; K)$ is of the form $y \mapsto h(y) = H(f(y))$, where $H \in C(f(Y); K)$. It is enough, then, to observe that the polynomial functions on $f(Y)$ are dense in $C(f(Y); K)$ by Mergelyan's Theorem, since the compact set $f(Y)$ does not separate the plane. (See item 26.4.3, page 226, in Čech, E., Point Sets, Academic Press Inc., 1969).
An analogous argument also establishes Theorem 4 in Rudin [12].

## § 4. BISHOP'S THEOREM AND ITS EXTENSION

The following result is proved in Bishop [1]. It refers to a unitary and closed subalgebra $A$ of $C(X; C)$. See also Glicksberg [4].

THEOREM: There exists a partition $P$ of $X$ into disjoint closed sets such that

(i) for each $S$ in $P$ the restriction $A_S$ of $A$ to $S$ is anti-symmetric,
(ii) if a function $f$ in $C(X; C)$ has, for each $S$ in $P$, a restriction to $S$ which belongs to $A_S$, then $f$ is in $A$.

The elements of $P$ are the maximal $A$-anti-symmetric sets (Glicksberg [4]). It is clear that the above theorem is a special case of our next result. The conventions of § 2 are still in force.

THEOREM 2: Let $G$ be a vector subspace of $C(X; E)$ which is a module over the unitary subalgebra $A$ of $C(X; K)$. Fix $f$ in $C(X; E)$. There is a maximal set of anti-symmetry for $A$, $U$, such that

$$\inf\{\|f-g\|;\ g \in G\} = \inf\{\|(f-g)|U\|;\ g \in G\}.$$

REMARK 6: Again, this equality is a sharpening of the relation

$$\inf\{\sup\{\|(f-g)|Y\|;\ Y \in A(X)\};\ g \in G\} = \\ = \sup\{\inf\{\|(f-g)|Y\|;\ g \in G\};\ Y \in A(X)\},$$

which now follows as a corollary.

PROOF: Theorem 1 and Zorn's Lemma will be used. Put

$$d = \inf\{\|f-g\|;\ g \in G\}.$$

Assume $d>0$: if $d=0$, for any

$$Y \in B(X),\ 0 \leqslant \inf\{\|(f-g)|Y\|;\ g \in G\} \leqslant \inf\{\|f-g\|;\ g \in G\} = d = 0.$$

Let $D$ be the set of all ordered pairs $(P, S)$ such that:

(i) $P$ is a partition of $X$ into non-empty pairwise disjoint and closed subsets of $X$;
(ii) $S$ is an element of $P$ such that $d = \inf\{\|(f-g)|S\|;\ g \in G\}$.

The pair $(\{X\}, X)$ belongs to $D$ which is therefore nonempty. Partially order $D$ by requiring $(P, S) \leqslant (Q, T)$ if, and only if, the partition $Q$ is finer than $P$ and $T \subset S$.

Let $C$ be a chain in $D$. An upper bound $(Q, T)$ of $C$ in $D$ is built as follows. For any $x \in X$ and $(P, S) \in C$ call $P(x)$ the one element of $P$ which contains $x$. Define $Q(x) = \bigcap\{P(x);\ (P, S) \in C\}$. The closed set $Q(x)$

is non-empty since $x \in Q(x)$. It is also clear that, for $x$ and $y$ in $X$, $Q(x) \cap Q(y)$ non-empty implies $Q(x) = Q(y)$. Therefore, the set

$$Q = \{Q(x);\, x \in X\}$$

is a partition of $X$ into pairwise disjoint compact subsets. Let

$$T = \bigcap \{S;\, (P, S) \in C\}.$$

To prove that $(Q, T) \in D$ and that it is an upperbound of $C$ in $D$ it is clearly enough to show that $T$ is non-empty and that

$$d = \inf\{\|(f-g)|T\|;\, g \in G\}.$$

Consider $\varepsilon > 0$ such that $d - \varepsilon > 0$. For each $g \in G$ and $(P, S) \in C$ the sets $K(g) = \{x \in T;\, \|f(x) - g(x)\| \geqslant d - \varepsilon\}$ and

$$K(g, P, S) = \{x \in S;\, \|f(x) - g(x)\| \geqslant d - \varepsilon\}$$

are compact and $K(g) \subset K(g, P, S)$. In fact, as it is easily seen,

$$K(g) = \bigcap \{K(g, P, S);\, (P, S) \in C\}.$$

Also, $K(g, P, S)$ is non-empty because $\inf\{\|(f-h)|S\|;\, h \in G\} = d$. It follows that $K(g)$ is non-empty. Indeed, otherwise, by the finite intersection property and because $C$ is a chain in $D$, there are $(P_1, S_1) \leqslant \ldots \leqslant (P_n, S_n)$ in $C$ such that $\bigcap_{j=1}^{n} K(g, P_j, S_j) = K(g, P_n, S_n) = \emptyset$, which is impossible. Since $T \supset K(g)$ we conclude that $T$ is non-empty. It remains to observe that $\inf\{\|(f-g)|T\|;\, g \in G\} \geqslant \inf\{\|(f-g)|K(g)\|;\, g \in G\} \geqslant d - \varepsilon$; $\varepsilon > 0$ being arbitrary and the inequality $\inf\{\|(f-g)|T\|;\, g \in G\} \leqslant d$ obvious, there follows the desired equality $d = \inf\{\|(f-g)|T\|;\, g \in G\}$. It has been proved that $(Q, T)$ is an upper-bound of $C$ in $D$.

By Zorn's Lemma there is a maximal element $(P, S) \in D$. We claim that $S$ is $A$-anti-symmetric.

Indeed, let $A_S$ be the set of the elements of $A$ which are real on $S$. By contradiction, admit that the unitary real subalgebra $B = A_S|S$ of $C(S, K)$ contains non-constant functions. Then the partition $B(S)$ of $S$ is distinct from $\{S\}$ (Remarks 1 and 3). Also, $G|S$ is a module over $B$. By Corollary 1 above, there is $T \in B(S)$, $T \neq S$, such that

$$d = \mathit{inf}\{\|(f-g)|S\|;\, \mathrm{g} \in G\} = \inf\{\|(f-g)|T\|;\, g \in G\}.$$

The partition $Q$ of $X$ consisting of the elements of $P$ distinct from $S$ and by the elements of $B(S)$ is strictly finer than $P$. Therefore, $(Q, T)$ is an element of $D$ such that $(P, S) < (Q, T)$ which contradicts the maximality of $(P, S)$. Let now $U$ be the maximal anti-symmetric set for $A$ which contains $S$. This ends the proof.

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REFERENCES

1. Bishop, E. – A generalization of the Stone-Weierstrass theorem, Pacific J. Math. **11**, 777–783 (1961).
2. Browder, A. – Introduction to function algebras, W. A. Benjamin, Inc., New York-Amsterdam (1969).
3. Buck, R. C. – Approximation properties of vector valued functions, Pacific J. Math. **53**, 85–94 (1974).
4. Glicksberg, I. – Measures orthogonal to algebras and sets of anti-symmetry. Trans. Amer. Math. Soc. **105**, 415–435 (1962).
5. Jewett, R. I. – A variation on the Stone-Weierstrass Theorem, Proc. Amer. Math. Soc. **14**, 690–693 (1963).
6. Leibowitz, G. M. – Lectures on complex function algebras, Scott, Foresman and Company (1970).
7. Machado, S. and J. B. Prolla – An introduction to Nachbin spaces, Rend. Circ. Mat. Palermo **21**, 119–139 (1972).
8. Nachbin, L. – Weighted approximation for algebras and modules of continuous functions: real and self-adjoint complex cases, Annals of Math. **81**, 289–302 (1965).
9. Nachbin, L. – Elements of approximation theory, D. van Nostrand Co., Inc. (1967). Reprinted by R. Krieger Co., Inc. (1976).
10. Prolla, J. B. – Bishop's generalized Stone-Weierstrass theorem for weighted spaces, Math. Ann. **191**, 299–289 (1971).
11. Rudin, W. – Real and complex analysis, McGraw-Hill, New York (1966).
12. Rudin, W. – Subalgebras of spaces of continuous functions, Proc. Amer. Math. Soc. **7**, 825–830 (1956).